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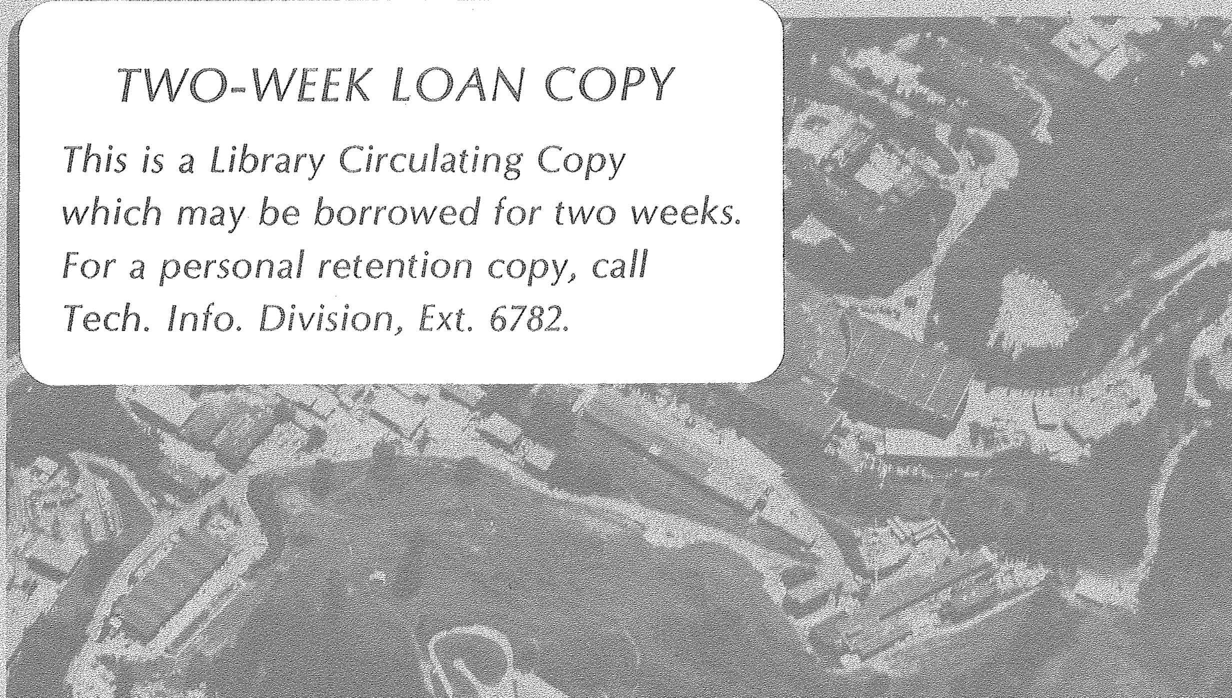

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GREEN'S FUNCTIONS OF VORTEX OPERATORS*

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ABSTRACT

We study the Euclidean Green's functions of the 't Hooft vortex operator, primarily for Abelian gauge theories. The operator is written in terms of elementary fields, with emphasis on a form in which it appears as the exponential of a surface integral. We explore the requirement that the Green's functions depend only on the boundary of this surface. The Dirac veto problem appears in a new guise. We present a two dimensional "solvable model" of a Dirac string, which suggests a new solution of the veto problem. The renormalization of the Green's functions of the Abelian Wilson loop and Abelian vortex operator is studied with the aid of the operator product expansion. In each case, an overall multiplication of the operator makes all Green's functions finite; a surprising cancellation of divergences occurs with the vortex operator. We present a brief discussion of the relation between the nature of the vacuum and the cluster properties of the Green's functions of the Wilson and vortex operators, for a general gauge theory. The surface-like cluster property of the vortex operator in an Abelian Higgs theory is explored in more detail.

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Introduction

Mandelstam[1,2] and 't Hooft[3,4] have shown that there is an electric/magnetic duality in the possible phases of non-Abelian gauge theories. A confining theory, for example, is dual to a completely broken Higgs theory. In the former, color-electric sources are confined, while in the latter, color-magnetic sources are confined. Just as a confining phase is characterized by an area law for the Wilson loop, a complete Higgs phase can be characterized by an area law for a dual operator, the vortex operator[3].

Strong restrictions on the possible phases of gauge theories have been obtained from this approach[4]. These results are essentially kinematic in nature: no one has found a way to rewrite a non-Abelian gauge theory as a simple dual field theory, and the idea of duality has not helped to answer the dynamical question of which phase is actually realized in any given non-Abelian gauge theory. This is in contrast to the case of U(1) and Z_N gauge and generalized gauge theories on a lattice, which can be transformed into dual gauge theories[6]; this duality is often a guide to the phase structure of a theory.

The results that have been obtained concerning the vortex operator are also kinematic, depending on its topological quantum number but not on its detailed form. In this paper we take a closer look at the details of the vortex operator. We deal primarily with an Abelian theory, leaving the complications of the non-Abelian theory for the future. In the next section we introduce the idea of the vortex operator, and express it in terms of elementary fields.

A problem arises which is essentially the veto problem from Dirac monopole theory. We study a two dimensional model of a Dirac string; the results suggest a new solution to the Dirac veto problem.

In section 3 we discuss the systematic expansion for the Green's functions of the vortex operator. We emphasize a form for the vortex operator in which it appears as the exponential of a surface integral, and we explore the condition that the Green's functions depend only on the boundary of the surface. We show how the Green's functions are expressed neatly in terms of Wu and Yang's idea of "sections". In section 4 we study the renormalization of two kinds of Abelian "looplike" operators, the Wilson loop and the vortex operator. In each case the possible divergences are easily determined with the aid of the operator product expansion, and for both operators only an overall multiplicative renormalization is needed. In the case of the vortex operator this result comes about through cancellation of a field renormalization divergence against a composite operator divergence.

In section 5 we discuss the relation between the Green's functions of the Wilson and vortex operators and the nature of the vacuum. We emphasize the cluster properties of Green's functions rather than the vacuum expectation value. We then show that a vortex operator in a massive theory always has surface-like clustering, and we see how this would appear from a graphical expansion. Section 6 presents a summary and discussion of the results.

2. Vortex Operators

The vortex operator, like the Wilson loop, is associated with a closed curve in spacetime. Let us consider a general "local loop-like" operator L associated with a closed curve C lying in the R^3 plane $t=t_o$, and let us consider its commutation relations with other operators at time t_o. By definition, L commutes with observable (that is, gauge invariant) local operators associated with a point $\vec{x}$ not on C. This implies that for any gauge dependent field $\Phi(\vec{x})$

$$L\Phi(\vec{x}) = \Phi(\vec{x})^g L \tag{2.1}$$

where $\Phi(\vec{x})^g$ is Φ after a gauge transformation $g(\vec{x})$ and $g(\vec{x})$ is some gauge transformation associated with the operator L. For instance, in the usual gauges (axial, covariant, Coulomb), the gauge transformation associated with the Wilson loop is simply 1 everywhere.

It is important that $g(\vec{x})$ is not defined in all of space, R^3, but only in $R^3 - C$. $R^3 - C$ is multiply connected: curves are distinguished by their winding number around C. As a result there can be a non-trivial effect, a topology, associated with $g(\vec{x})$. As $\vec{x}$ describes a path around C, $g(\vec{x})$ describes a path through the gauge group. If the gauge group is simply connected, there is nothing interesting about this, but if it is multiply connected the path may lie in a non-trivial element of $\Pi_1(G)$. Homotopy groups (Π_n) are discussed in reference 8. For different paths of the same winding number, continuity requires that this element be the same. Since paths of winding number other than one can be generated as a product of paths of winding number one (traversed backwards for negative winding number) the homotopy class associated with winding number one determines that for any other path. The class associated with L is gauge invariant: since any gauge transformation can be continuously deformed into one which is unity in an arbitrarily large volume (including the whole of L) the gauge transformed $g(\vec{x})$ can be continuously deformed into its original value; homotopic invariants are therefore unchanged. Thus, there is a gauge invariant quantum number, a homotopy class, associated with any looplike operator.

Operators for which this class is not the trivial one are called vortex operators. When such an operator acts on the vacuum of a completely broken Higgs theory, it produces the twisted boundary conditions associated with the Nielsen-Olesen vortex[9]. The operator creates this vortex state; hence the name.

In an Abelian gauge theory, a vortex operator creates a loop of magnetic flux just as the Wilson operator creates a loop of electric flux. In the dual (Abelian) gauge theories mentioned earlier, the Wilson loops of one theory are mapped into vortex operators of the other. In the long distance, large coupling, limit of non-Abelian (as well as Abelian) gauge theories, the vacuum approaches an eigenstate of a simple vortex operator. Thus, they are attractive operators to consider. If one tries instead to consider duality in terms of pointlike, monopole creation, operators, one finds that there is no associated topological quantum number: $\Pi_2(G)$ is trivial for any Lie gauge group G. In this, there is an interesting analogy between electric and magnetic quantities: the pointlike operators in non-Abelian theories (gauge fields or monopoles) have no gauge or topological invariance, while the looplike operators (Wilson loops or vortices) do.

The quantum number associated with a vortex can be characterized in a different way. Consider the path described by $g(\vec{x})$ as $\vec{x}$ winds once around C. This path in G defines in a natural way $(g^{-1}\vec{\partial}g=\tilde{g}^{-1}\vec{\partial}\tilde{g})$ a path in the simply connected covering group $\tilde{G}$. By a well-known connection between $\Pi_1(G)$ and the center of $\tilde{G}$, when $\vec{x}$ returns to its starting position $\tilde{g}(\vec{x})$ need not return to its original value but is multiplied by an element Z_L of the center of $\tilde{G}$. Because $g(\vec{x})$ is single valued, Z_L in $\tilde{G}$ must be mapped into 1 in G by the usual homomorphism. Vortex operators can thus be considered to have quantum numbers in $Z(\tilde{G})/Z(G)$, the quotient of the centers of the two groups, which is isomorphic to $\Pi_1(G)$. It also follows that if r is a representation single valued in G, $z_{rL}=1$, where z_{rL} is Z_L in representation r.

One can show from equation 2.1 that

$$L(C)W_s(C') = W_s(C')L(C)(z_{sL})^{w(C,C')} \qquad (2.2)$$

where $w(C,C')$ is the winding number of C' through C, and $W_s(C')$ is the Wilson loop in representation s associated with curve C' (assumed here to lie in the $t=t_o$ hyperplane). Thus, although L(C) commutes with every <u>local</u> gauge invariant not on C, if it is a vortex operator it will not commute with certain Wilson loops linking C. Equation 2.2 characterizes completely the topological character of the vortex operator; it is only Z_L (or z_{sL} for all s), not $g(\vec{x})$ that can be defined in a gauge invariant way. Note that equation 2.2 is entirely dual between L(C) and $W(C')$; one cannot say that one is a topological operator and the other is not, until one tries to discuss gauge dependent quantities as in equation 2.1. The one genuine asymmetry is that the Hamiltonian is relatively simple in terms of the Wilson loops (or the related vector potentials), but does not appear to have a simple form in terms of vortex operators; this asymmetry may or may not be permanent.

We illustrate these ideas for a U(1) theory. The covering group of U(1) is R^1; a general element of R^1 is a real number y. In the representation of charge e, y becomes exp(iey). A general looplike operator L(C) is then associated, through equation 2.1, with a function $y(\vec{x})$, the R^1 version of $\tilde{g}(\vec{x})$. Like $\tilde{g}(\vec{x})$, $y(\vec{x})$ need not be single-valued, but $\exp\{iey(\vec{x})\}$ must be if fields of charge e are present, because $g(\vec{x})$ is single-valued. It follows that when $\vec{x}$ winds once around C, $y(\vec{x})$ must change by $2\pi p/e_{min}$, where e_{min} is the unit of charge and p is any integer. Vortex operators are thus characterized by an integer p. Equation 2.2 is now

$$L(C)W_q(C') = W_q(C')L(C)\exp\{ipqw(C,C')/e_{min}\} \qquad (2.3)$$

where W_q is the Wilson loop

$$W_q(C') = \exp\{iq\oint_{C'}dx_i A_i(\vec{x})\} \qquad (2.4)$$

For those fields actually present, q is a multiple of e_{min} and the phase factor in equation 2.3 is 1.

Differentiating equation 2.3 with respect to q and setting q=0, and then using Stokes's theorem to relate the line integral of $\vec{A}$ to the surface integral of the magnetic field $\vec{B}$,

$$L(C)B_i(\vec{x}) = \{B_i(\vec{x}) + \frac{2\pi ip}{e_{min}}\oint_C dy_i \delta^3(\vec{x}-\vec{y})\}L(C) \qquad (2.5)$$

Although equation 2.5 refers to a commutator directly on C, it follows from equation 2.3 and is therefore true for any vortex operator independent of its short distance details. This is the source of the statement that vortex operators create a loop of magnetic flux; the dual equation, replacing L by W_e, $\vec{B}$ by $\vec{E}$ and $2\pi p/e_{min}$ by e is also true. There is no corresponding local version of equation 2.4 for a non-Abelian theory. Equation 2.2 might then be taken as a definition of non-Abelian magnetic flux[3].

An operator satisfying equation 2.3 is

$$V_p(C) = \exp\{\frac{ip}{e_{min}}\int d^3x\ \frac{\hat{\theta}}{\rho}\cdot\vec{E}(\vec{x})+\theta j_0(\vec{x})\} \tag{2.6}$$

where θ,ρ, and z are cylindrical coordinates, $\hat{\theta}$ is a unit vector in the θ direction, and j_0 is the charge density. We take the $A_0=0$ gauge for convenience, but V(C) is gauge invariant and so will be its commutators, such as 2.1, with gauge invariant operators. From the canonical commutators

$$[E_i(\vec{x}),A_j(\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{2.7a}$$

$$[j_0(\vec{x}),\psi(\vec{y})] = -e_\psi\delta^3(\vec{x}-\vec{y})\psi(\vec{y}) \tag{2.7b}$$

one finds

$$V_p(C)\psi(\vec{x}) = \exp[\frac{-ip\theta e_\psi}{e_{min}}]\psi(\vec{x})V_p(C) \tag{2.8a}$$

$$V_p(C)A_i(\vec{x}) = \{A_i(\vec{x}) + i\exp\{-ip\theta/e_{min}\}\partial_i\exp\{ip\theta/e_{min}\}\}V_p(C) \tag{2.8b}$$

$$V_p(C)E_i(\vec{x}) = E_i(\vec{x})V_p(C) \tag{2.8c}$$

This is the Abelian form of equation 2.1. The curve C is here the z-axis; $g(\vec{x})$ is $\exp[-ip\Theta]$. A vortex operator for any C and any $g(\vec{x})$ can be constructed in the same way.

The coordinate Θ must have a discontinuity of 2π on a semi-infinite surface S bounded by the z-axis (such as the half-plane $\Theta=0=2\pi$). The exponent in the definition of $V_p(C)$ is therefore discontinuous but the operator itself has no discontinuity on the surface, as can be seen from its commutators. These are completely independent of where we choose to define the discontinuity of Θ. Using

$$\partial\Theta = \frac{\hat{\Theta}}{\rho} + \mathrm{disc}(\Theta)$$

$V_p(C)$ can be rewritten

$$V_p(C) = \exp\{\frac{ip}{e_{min}}[\int d^3x\ (j_0(\vec{x})-\partial_i E_i(\vec{x}))\Theta + 2\pi\int_S dn_i E_i\ ]\} \qquad (2.9)$$

$j_0(\vec{x})-\partial_i E_i(\vec{x})$ does not vanish as an operator in the $A_0=0$ gauge, but by Gauss's law it vanishes in gauge invariant Green's functions. The gauge invariant Green's functions are therefore the same for $V_p(C)$ as for

$$V_p{}'(C) = \exp\{\frac{2\pi ip}{e_{min}}\int_S dn_i E_i\} \qquad (2.10)$$

If one evaluates the commutator of $V_p{}'(C)$ with $\Theta(\vec{x})=\psi^*(\vec{x})[\partial_i-ie_\psi A_i(\vec{x})]\psi(\vec{x})$, it does not appear to vanish on S; it must, however, because $[V_p(C),\Theta(\vec{x})]=0$ on S and Θ is gauge invariant. The problem is that $V'(C)$ is too singular for canonical commutators to be correct; if one evaluates the Green's functions of $V_p{}'(C)\Theta(x)$ using the methods of the next section, one finds that they have no equal time discontinuity on S. $V_p{}'(C)$ is a more convenient form of the operator when one is discussing Green's functions.

One might wonder whether the commutators 2.8 are really correct in field theory, even for the form 2.9, or whether some anomaly will develop due to the discontinuity of θ. V and V′ are poorly defined because the discontinuity of θ is sharp; if we define them by smearing the discontinuity and taking the smearing to zero, does the limiting operator satisfy 2.8? A simple example shows this to be a valid worry. Consider a theory with fermions, ψ, in two spacetime dimensions (they may even be free fermions), and consider the operator

$$\Omega(x) = \exp\{\int_{x_1}^{\infty} dx'_1 \; 2\pi i \bar{\psi}(x')\gamma_0\psi(x')\} \tag{2.11}$$

where x is the spacetime point (x_0,x_1) and the integral runs along the equal time path from (x_0,x_1) to (x_0,∞). From the commutator

$$[\bar{\psi}(y)\gamma_0\psi(y),\psi(z)]\delta(x_0-y_0) = -\delta^2(y-z)\psi(y) \tag{2.12}$$

we find, by the same canonical manipulations that lead from 2.7b to 2.8a, that $[\Omega(x),\psi(z)]=0$ at equal times, so that $\Omega(x)$ would be a c-number. On the other hand, from boson equivalence[10],

$$\begin{aligned}\Omega(x) &= \exp\{\int_{x_1}^{\infty} dx'_1 \; -2\sqrt{\pi} i \partial_1 \phi(x')\} \\ &= \exp\{2\sqrt{\pi} i \phi(x)\} \\ &= \text{const}\cdot\bar{\psi}(x)(1+\gamma_5)\psi(x)\end{aligned} \tag{2.13}$$

If the fermions are massive, the leading piece of $\bar{\psi}(1+\gamma_5)\psi$ is in fact a c-number, but if they are massless, $\bar{\psi}(1+\gamma_5)\psi$ has no c-number piece and $[\Omega,\psi]$ is not identically zero.

One may also see this in a different way by evaluating

$$G(x,y,z) = < \Omega(x)\, \bar{\psi}(y)\, \psi(z) > \qquad (2.14)$$

for free fermions. $G(x,y,z)$ is poorly defined because the support in the exponent in equation 2.11 is too singular. If one smears the support (necessarily into the time direction) over a small distance λ (call the resulting operator Ω_λ and the Green's function G_λ), we can calculate $G_\lambda(x,y,z)$ directly. As λ goes to zero, for free massive fermions $G_\lambda(x,y,z)$ approaches $< \bar{\psi}(y)\, \psi(z) >$ times a constant, while for massless fermions it approaches

$$< \bar{\psi}(x)(1+\gamma_5)\psi(x)\, \bar{\psi}(y)\, \psi(z) >$$

times a constant.

This is not a serious problem. By regulating Ω in a slightly more complicated way, one may obtain the desired limit. Consider the operator

$$\bar{\psi}(x)(1+\gamma_5)\psi(x)\Xi_\lambda(x)$$

where

$$\Xi_\lambda(x) = \exp\{i\int d^2z\ f_\lambda(z)\bar{\psi}(x+z)\psi(x+z)\} \qquad (2.15)$$

and $f_\lambda(z_0,z_1)$ is a family of functions with support in $|z| < \lambda$. If f_λ is defined so that as $\lambda \to 0$,

$$\int d^2z\ f_\lambda(z)/|z|^2 \to \infty \qquad (2.16a)$$

$$\int d^2z\ f_\lambda(z)/|z| \to 0 \qquad (2.16b)$$

then as $\lambda \to 0$, $\bar{\psi}(1+\gamma_5)\psi\,\Xi_\lambda$ approaches a c-number even for massless fermions. If one then defines a regulated $\Omega(x)$ as $\Omega_\lambda(x)\Xi_\lambda(x)$, its limit will be a c-number. Equations 2.16 are correct for free fermions or with a super-renormalizeable interaction; as one might expect, with a Thirring interaction there would have to be different powers of z in the integrands.

It is interesting to repeat some of the above analysis for the Schwinger model, without using bosonisation. Using Gauss's law, we get

$$\Omega_\lambda(x) = \exp\{-\frac{2\pi i}{e}\int d^2z\ \delta_\lambda^2(z)E(x+z)\} \tag{2.17}$$

where $\delta_\lambda^2(z)$ is the delta function, smeared over a distance λ in spacetime, and E, the electric field, is a (pseudo-)scalar in two dimensions. The analogy between this and equation 2.10 is clear.

We would like to compare

$$G(c,y) = \int_c dx_\mu \epsilon_{\mu\rho} < j_{5,\rho}(x)\ \Omega_\lambda(y) > \tag{2.18}$$

with the same quantity with 1 in place of Ω_λ. Here $j_{5,\rho}$ is the axial current $\bar{\psi}\gamma_5\gamma_\rho\psi$ and c is a spacetime curve circling the spacetime point y at some distance large compared to λ but small compared to $1/e^2$ (which has dimensions of length). Because of the latter stipulation, we can neglect all but the leading term from perturbation theory.

Using the anomalous divergence equation,

$$\partial_\mu j_{5,\mu}(x) = -\frac{e}{\pi}E(x) \tag{2.19}$$

we can rewrite equation 2.18 as

$$G(c,y) = \frac{e}{\pi}\int_r d^2x < E(x)\ \Omega_\lambda(y) > \tag{2.20}$$

From

$$< E(x)\ E(y) > = \delta^2(x-y) + O(e^2) \tag{2.21}$$

it follows that

$$< E(x)\ \Omega_\lambda(y) > = -\frac{2\pi i}{e}\ \delta_\lambda^2(x-y)<\Omega_\lambda(y)> + O(e) \tag{2.22}$$

and

$$G(c,y)\ /\ <\Omega_\lambda(0)> = -2 + O(e^2) \tag{2.23}$$

When Ω_λ is replaced by a c-number in 2.18 and 2.23, the ratio 2.23 is zero: the limit of Ω_λ as λ goes to zero is not a c-number. Ω has much in common with a Dirac string; equation 2.22 shows that there is a lump of flux which does not go away when λ goes to zero. Our result here is that a massive fermion does not feel this flux in the limit, whereas a massless fermion does, but that a massless fermion can be "shielded" from the flux by the additional regulator 2.15. Massive and massless bosons appear to behave like massive fermions: the Green's function analogous to 2.14 goes to the free propagator in the limit.

Our results for the four dimensional case are not so complete. We discuss them, and more of the analogy with the Dirac string, in the next section.

3. Green's Functions of Vortex Operators

In this section we will discuss the calculation of Green's functions of the $V'(C)$ form of the vortex operator in Euclidean space-time. For simplicity we shall assume only one charged field, scalars $\phi(x)$ with charge e; the generalization to more fields and to fermions is straightforward. We can now consider a vortex operator associated with a general closed curve in spacetime. By analogy to equation 2.10 we define

$$V_p'(C,S) = \exp\{\frac{\pi p}{2e}\int_S d\sigma^y_{\alpha\beta} F_{\gamma\delta}(y)\epsilon_{\alpha\beta\gamma\delta}\}$$

$$=\exp\{\frac{\pi p}{e}\int_S d\sigma^y_{\alpha\beta}\tilde{F}_{\alpha\beta}(y)\} \tag{3.1}$$

where S is any 2-surface whose boundary is C and $\epsilon_{1234}=1$. There is no factor of i in the exponent because $F_{j4}=iE_j$ in Euclidean spacetime. To show that 3.1 is correct, we would like to check that it satisfies equation 2.3, and that its gauge invariant Green's functions are independent of the surface S.

Consider now a general Green's function of V_p':

$$< V_p'(C,S) >_{JKL} = \tag{3.2}$$

$$< V_p'(C,S)\ \exp\{\int d^4x J_\mu(x)A_\mu(x)+K(x)\phi(x)+L(x)\phi^*(x)\} >$$

where J_μ, K, and L are general external sources. The expansion of V_p' involves terms with arbitrarily large powers of 1/e, but owing to the exponential form of V′, these sum up in a convenient way:

$$< V_p'(C,S) >_{JKL} = \exp \sum_{khmn} \frac{1}{k!h!m!n!} \tag{3.3}$$

$$< [\frac{\pi p}{e}\int_S d\sigma^y_{\alpha\beta}\tilde{F}_{\alpha\beta}(y)]^k [\int d^4x\ J_\mu A_\mu]^h [\int d^4x\ K\phi]^m [\int d^4x\ L\phi^*]^n >_c$$

where $< >_c$ indicates the connected Green's function. The connected Green's function is of order at least e^{k+h} (e being the charge), except for k+h=2, m=n=0, for which it starts at order e^0. The sum in 3.3 therefore starts at order e^{-2}. Further, the graphs contributing at each order in e are readily classified: a graph with n_i internal photon lines (internal means both ends connected to charged lines) and n_e external lines (one end attached to a charged line and one attached to J_μ) is of order $e^{2n_i+n_e}$, independent of the number of vortex photon lines (defined as lines which run from S, the vortex operator, to a charged line). The connected graph with no charged lines (one photon line with both ends attached to S) is of order e^{-2}.

For example, the h=m=n=0 term, which is independent of J_μ, K, and L, and is the only term which contributes to $\langle V_p'(C,S)\rangle$, is

$$\langle V_p'(C,S)\rangle = \exp\{\frac{2\pi^2p^2}{e^2}\delta^2(0)A(S) \tag{3.4}$$

$$-\frac{p^2}{2e^2}\oint_C dx_\mu \oint_C dy_\mu (x-y)^{-2} - \ln\det\{\Delta^{vS}/\partial^2\} + O(e^2)\}$$

where A(S) is the area of the surface S, Δ^{vS} is $\{D_\mu^{vS}D_\mu^{vS}\}^{-1}$, D_μ^{vS} is $\partial_\mu - ieA_\mu^{vS}(x)$ and $A_\mu^{vS}(x)$ is given by the graph of figure 1a. The first two terms are from the one photon graph. The third is the sum of all graphs with one charged line connected to S by any number of photon lines. The first term is S dependent and quadratically divergent. We shall postpone further discussion of 3.4 until the end of this section. We will consider until then "reduced" Green's functions, with the h=m=n=0 term divided out.

To investigate the dependence on S of gauge invariant reduced Green's functions, consider first the order e^0 term of $\langle F_{\mu\rho}(x)\, V_p'(C,S)\rangle / \langle V_p'(C,S)\rangle$, which is equal to

$$\partial_\mu A_\rho^{vS}(x) - \partial_\rho A_\mu^{vS}(x) = F_{\mu\rho}^{vS}(x) \tag{3.5}$$

A^{vS}_{μ} was defined above; 3.5 defines $F^{vS}_{\mu\rho}$ as its curl.

$$F^{vS}_{\mu\rho}(x)-F^{vS'}_{\mu\rho}(x) = \frac{\pi p}{2e}\int_{S-S'} d\sigma^{y}_{\alpha\beta}\epsilon_{\alpha\beta\gamma\delta} < F_{\gamma\delta}(y)\ F_{\mu\rho}(x) >_{o}$$

$$= -\frac{\pi p}{e}\int_{S-S'} d\sigma^{y}_{\alpha\beta}[\ \epsilon_{\alpha\beta\gamma\rho}\partial^{y}_{\mu} + \epsilon_{\mu\alpha\beta\gamma}\partial^{y}_{\rho}\ ]\partial^{y}_{\gamma}\Delta_{E}(x-y) \qquad (3.6)$$

where $< >_{o}$ indicates the free propagator and Δ_{E} is the Euclidean scalar propagator:

$$\partial^{2}\Delta_{E}(x-y)=-\delta^{4}(x-y) \qquad (3.7)$$

Using the fact that a completely antisymmetric 5-tensor vanishes in four dimensions, the quantity in square brackets can be rewritten as

$$\epsilon_{\rho\mu\alpha\beta}\partial^{y}_{\gamma} + \epsilon_{\gamma\rho\mu\alpha}\partial^{y}_{\beta} + \epsilon_{\beta\gamma\rho\mu}\partial^{y}_{\alpha}$$

After surface integration the last two terms vanish, because S-S′ has no boundary, while the remaining term gives, using 3.7,

$$F^{vS}_{\mu\rho}(x)-F^{vS'}_{\mu\rho}(x) = -\frac{\pi p}{e}\int_{S-S'} d\sigma^{y}_{\alpha\beta}\epsilon_{\alpha\beta\mu\rho}\delta^{4}(x-y)$$

$$= -\frac{2\pi p}{e}\int_{S-S'} d\tilde{\sigma}^{y}_{\mu\rho}\delta^{4}(x-y) \qquad (3.8)$$

We see that the Green's function is S-dependent, but only when x lies directly on S or S′. However, when x lies on S, $< F_{\mu\rho}(x)\ V_{p}'(C,S) >$ is not well defined due to problems of operator ordering. If we resolve this ordering problem by taking a limit as x approaches S, $F^{vS}_{\mu\rho}$ is S independent everywhere. In effect this is taking the T-product, whereas 3.5 defines the T*-product (the covariant T* product is discussed in reference 11).

$$T[F^{v}_{\mu\rho}(x)] = T^{*}[F^{vS}_{\mu\rho}(x)] + \frac{2\pi p}{e}\int_{S} d\tilde{\sigma}^{y}_{\mu\rho}\delta^{4}(x-y) \qquad (3.9)$$

is independent of S.

Now let us go one order further, and consider

$$G(P,C,S) = \tag{3.10}$$

$$< \phi^*(x)\exp\{ie\int_P dx'_\mu A_\mu(x')\}\phi(y)\ V_p'(C,S) > / < V_p'(C,S) >$$

where P is some path from y to x. To lowest (e^0) order, this gauge invariant Green's function is given by all graphs of the form shown in figure 1b. These graphs sum up to give

$$G(P,C,S) = -\Delta^{vS}(y,x)\exp\{ie\int_P dx'_\mu A^{vS}_\mu(x')\} \tag{3.11}$$

Using the path integral representation[12] for the propagator $\Delta^{vS}(y,x)$, we can write

$$G(P,C,S) = -\exp\{ie\int_P dx'_\mu A^{vS}_\mu(x')\} \tag{3.12}$$

$$\int \overline{Dz}\ ds\ \exp\{\int_0^s dt - \frac{m^2}{2} - \frac{1}{2}\dot{z}_\mu\dot{z}_\mu + ie\dot{z}_\mu A^{vS}_\mu(z)\}$$

with $\overline{Dz}$ indicating the integral over all paths z(t) such that z(0)=x and z(s)=y, and $\dot{z}_\mu$ is dz_μ/dt. As long as A^{vS}_μ is smooth (we must temporarily smear out the surface S) the path integral can be made mathematically precise (see the review article by Gelfand and Yaglom[13]); at this level we simply have an external field problem and there are no short distance difficulties. The right hand side of 3.12 depends on A^{vS}_μ only through the phase factors

$$\exp\{ie\oint_{C'} dx'_\mu A^{vS}_\mu(x')\} = \exp\{\frac{ie}{2}\int_{R(C')} d\sigma_{\alpha\beta} T^*[F^{vS}_{\alpha\beta}]\} \tag{3.13}$$

where C' is the closed curve formed by joining P with z(t) and R(C') is any 2-surface whose boundary is C'. To obtain equation 3.13 we had to use equation 3.1, which defines the S-dependent $T^*[F^{vS}_{\mu\rho}]$. We cannot try to simply define this S-dependence away by replacing $T^*[F^{vS}_{\alpha\beta}]$ with $T[F^{v}_{\alpha\beta}]$, as the result would in general depend on the choice of the arbitrary surface R.

From 3.8 and 3.13, it follows that changing the surface on which V_p' is defined changes the phase associated with the path C′ by a factor

$$\exp\{i\pi p\int_{S-S'} d\sigma^x_{\alpha\beta}\int_R d\tilde{\sigma}^y_{\alpha\beta}\delta^4(x-y)\} \tag{3.14}$$

In four dimensions, a closed curve, C′, links a closed 2-surface, S-S′, a definite, integral, number of times, N_I (leaving out for now those particle paths which actually intersect S or S′). Any surface R bounded by C′ will then intersect S-S′ at N_I points (intersections are defined in an oriented way, so that it is the net number of intersections that is counted). Setting up local coordinate systems on S-S′ and on R near such an intersection, one finds that the double integral in 3.14 is exactly $2N_I$. The phase then changes by $\exp\{2\pi i p N_I\}$ under the change of surface, which is 1 if p is an integer as required by the earlier quantization condition.

The above argument is extremely familiar: it is just the argument that one can have magnetic monopoles in an Abelian gauge theory, if the monopole charge is quantized so that the charged particles don′t "see" the monopole′s Dirac string[7]. In fact, just as the Wilson loop can be regarded as the world-line of a classical charged particle, the vortex operator can be regarded as the world-line of a classical monopole, with the surface S as the world-sheet of the Dirac string. There is one problem with the Dirac string, and that problem is also present here: what of paths which actually pass through the surface S? If we smear the integral defining V_p' (or that for V_p) and let the smearing vanish as a limit, can we neglect paths passing through S because they are of "measure zero", or do they contribute in a sufficiently singular way that their effect does not go away?

This is exactly the question that was raised from a different point of view at the end of the last section. In our two dimensional model it was found, in effect, that when the support of our Dirac string (there it was a "Dirac lump") goes to zero with the total flux staying constant, under some conditions (massless fermions) the effect of paths through the lump survives in the limit, while under others (massive fermions and bosons) the effect vanishes. Even in the massless fermion case it was possible to obtain the desired limit by adding additional regulators to the operator, which can be thought of as correcting the action for paths that pass through the lump.

The four dimensional case is not so easily analyzed. For S an infinite 2-plane (so that C is infinitely far off in some direction) the charged field propagator is easily calculated with the flux smeared, and one finds that it approaches the free propagator for massive or massless fermions, or for bosons. In the presence of this 2-plane plus an additional smooth A_μ-field, however, an argument similar to that for the Schwinger model (based on the anomalous divergence of the axial current) shows that the effect of the Dirac string survives in the limit if the fermions are massless. An extra regulator of the same form as Ξ in 2.17 (essentially giving the fermions a mass very near the string) corrects this particular problem. We have not been able to show, however, that this is the only additional operator needed in general, or that no correction is ever needed for massive fermions or bosons. It seems quite likely, however, that if we do need to correct the action for paths passing through the string, this correction will always take the form of local operators near the string. The whole point is, perhaps, moot, as we shall see that the Green's functions are uniquely determined without knowing the detailed form of the extra regulator (just as in the two dimensional case, we do not really need to know the form of Ξ; once we know that Ω is a c-number, all of its Green's functions are fixed up to an overall constant!).

Many other solutions for the Dirac veto problem have been given. Brandt, Neri, and Zwanziger[7] have rewritten a field theory of charges and monopoles as an integral over all numbers and configurations of particle paths. They then define the action associated with configurations having charge paths intersecting monopole strings as the limit of that for non-intersecting configurations. In a monopole field theory, our solution would have the form of a smeared Dirac string and an additional non-local charge monopole interaction along the string.

Hereafter we assume that $V_p'(C,S)$ has been defined in such a way as to have the desired limit, so that $G(P,C,S)$ is independent of S. It then follows that there is a gauge transformation $g(S,S';x)$, defined except when x lies on S or S', such that

$$\Delta^{vS'}(y,x) = g(S,S';y)\Delta^{vS}g^*(S,S';x) \tag{3.15a}$$

$$A_\mu^{vS'}(x) = A_\mu^{vS}(x) + \frac{i}{e}g(S,S';x)\partial_\mu g^*(S,S';x) \tag{3.15b}$$

Any gauge independent quantity constructed from A_μ^{vS} and Δ^{vS} is therefore S-independent as well.

At least in the present case, where there is only a classical monopole, we can evaluate quantities such as G(P,C,S) or $\Delta^{vS}(x,y)$ without using either an explicit form for $V_p'(C,S)$ or the path integral prescription. Equations 3.15 by themselves give sufficient information to determine the Green's functions we want, using a geometrical approach (in the sense of differential geometry) due to Wu and Yang[14]. Consider two non-intersecting surfaces, S_1 and S_2, each having C as its boundary. Take spacetime with $\tilde{C}$, a thin tube containing C, removed. This space can be covered by two overlapping open regions, Q_1 and Q_2, such that S_1 lies entirely in Q_2, and S_2 lies entirely in Q_1. $A_\mu^{vS_1}(x)$ is defined for x in Q_1, and $A_\mu^{vS_2}(x)$ is defined for x in Q_2; in the overlap region these functions are related by 3.15b. They are given, as before, by the simple one photon graph, and $g(S_1,S_2;x)$ can be determined in terms of this graph. Any other gauge and S-dependent quantity can similarly be represented as a pair of functions (or, for Δ^v as four functions, since it has two arguments), each defined only in a certain region but related in the overlap region by the gauge transformation $g(S_1,S_2;x)$. Gauge dependent quantities are thus "sections": sets of functions each defined only in an open region, but such that the regions cover all of spacetime (minus $\tilde{C}$ in this case) and such that the functions are related in the overlap of two or more regions.

Integrals of gauge (and therefore S-) independent products of sections over $R^4 - \tilde{C}$ can then be defined: in each region the integrand is defined in terms of the function which exists in that region; in overlap regions this is unambiguous because the integrand is S-independent. With an inner product based on this integral, $D^{v}_{\mu}D^{v}_{\mu}$ acting on sections can be made self-adjoint. It follows that Δ^{v} as a section exists and is unique. Δ^{vS} can be obtained from it as a function defined everywhere except on S. Power counting arguments, as developed in the next section, indicate that the limit as the cutoff $\tilde{C}$ is removed exists, at least order by order in p.

We may now consider higher order corrections to Green's functions. All graphs with a given configuration of charged lines and internal and external photon lines are of the same order in e; summing over all numbers of vortex photon lines gives one graph of the same configuration with no vortex photon lines but with "effective" propagators and vertices. The photon propagator and $\phi^{*}\phi A^{2}$ vertex are unchanged, and the charged propagator becomes $-\Delta^{vS}(x,y)$. The $\phi^{*}\phi A_{\mu}$ vertex picks up an extra term from graphs where a $\phi^{*}\phi A^{2}$ vertex is connected to one vortex photon line and one other photon line; it becomes $-ie\overleftrightarrow{D}^{vS}_{\mu}(x)$. Equation 3.3 then becomes

$$< V_p{}'(C,S) >_{JKL} / < V_p{}'(C,S) > = \exp \sum_{hmn}{}' \frac{1}{h!m!n!} \tag{3.16}$$

$$< [\int d^4x \; J_\mu A_\mu]^h [\int d^4x \; K\phi]^m [\int d^4x \; L\phi^*]^n >_{cvS}$$

where the prime on the sum excludes the term h=k=l=0 and the subscript cvS indicates all connected graphs constructed out of the effective vertices and propagators. Examples of higher order corrections to $< A_\mu(x) \; V_p{}'(C,S) > / < V_p{}'(C,S) >$ are shown in figure 2.

It is then evident that higher order corrections to gauge invariant reduced Green's functions are S-independent: under a change of S, the phase factors from two propagators meeting at a $\phi^*\phi A^2$ vertex or an external source cancel; the total change in the propagators and vertex at a $\phi^*\phi A_\mu$ vertex vanishes. Actually, this is not strictly true if we have defined the vortex operator by smearing S and adding additional operators. The Feynman integrals in coordinate space include points lying in S, for which equations 3.15 do not hold. As in the case of the propagator, we would expect to be able to "repair" the Feynman integrals with additional corrections to the definition of $V_p'(C,S)$. Again, we need never know the form of these corrections: regarding the propagators and vertex functions as sections, the Feynman integral for each graph can be written as the invariant integral discussed above, and the result is unique.

We can now demonstrate equation 2.5:

$$\begin{aligned}\partial_\mu T\langle \tilde{F}_{\mu\rho}(x)\, V_p'(C,S)\rangle &= \partial_\mu (T-T^*)\langle \tilde{F}_{\mu\rho}(x)\, V_p'(C,S)\rangle \\ &= \partial_\mu (T-T^*)\langle \tilde{F}_{\mu\rho}(x)\, V_p'(C,S)\rangle_o \\ &= \frac{2\pi p}{e}\oint_C dy_\rho\, \delta^4(x-y)\langle V_p'(C,S)\rangle \qquad (3.17)\end{aligned}$$

The first equation follows because $T^*(\tilde{F}_{\mu\rho})$ is defined as $\epsilon_{\mu\rho\alpha\beta}\partial_\alpha T^*(A_\beta)$ and so its divergence vanishes identically. The second equality (the subscript indicates the lowest order graph, figure 1a) follows because any higher order graphs for the T^* Green's function, such those of figure 2, are S-independent and therefore continuous when x is at S: they do not contribute to $(T-T^*)$. The final equality follows from equation 3.9. Since $\partial_\mu \tilde{F}_{\mu\rho}(x)=0$ is true as an operator equation, 3.17 represents a commutator and is in fact the covariant version of 2.5. Equation 2.5 is an operator equation and we have only considered one matrix element of it; the same argument can be readily applied to any gauge invariant matrix element.

Equation 2.3 is also true. If we consider $< W_q(C') \; V_p{}'(C,S) >$, only the graph of figure 3a has a discontinuity whenever C' crosses S: the Green's function jumps by a factor $\exp[\frac{2\pi ipq}{e}]$. This is the covariant form of 2.3. Figures 3b, 3c, etc. are S-independent and therefore continuous at S. Equation 2.5 thus exponentiates to give 2.3. Ordinarily this would not be true, because in general the commutators of $\vec{B}$ do not determine those of the Wilson loop (although they do in naive canonical manipulations) because graphs such as 3c which depend on the composite nature of $W_q(C')$ have discontinuities; this is not a problem here. The location of the discontinuity of $< W_q(C') \; V_p{}'(C,S) >$ does depend on S, unlike the Green's functions of local gauge invariants.

For later use we would like to examine the S-dependence of gauge-dependent quantities. From 3.15 and the effective Feynman rules, it follows that

$$< A_\mu(x)\ldots\phi(y)\ldots\phi^*(z)\ldots V_p{}'(C,S') > / < V_p{}'(C,S') > \tag{3.18}$$

$$= < A_\mu(x)+\frac{i}{e}g(S,S';x)\partial_\mu g^*(S,S';x)\ldots g(S,S';y)\phi(y)\ldots$$

$$g^*(S,S';z)\phi^*(z)\ldots V_p{}'(C,S) > / < V_p{}'(C,S) >$$

We shall also be interested in the singularities of gauge dependent quantities near S. The singular behavior of $A_\mu^{vS}(x)$, all from figure 1a, is:

$$A_\rho^{vS}(x) \approx \frac{p}{e}\epsilon_{\alpha\beta\mu\rho} n_{1\alpha} n_{2\beta} r_\mu / r^2 \tag{3.19a}$$

where n_1 and n_2 are orthogonal unit vectors lying in S and r is the vector from x to the nearest point on S. From 3.15b and 3.19a we derive

$$g(S,S';x) \approx \exp[ip\Theta(x)] \tag{3.19b}$$

where S′ is any surface distant from S and x. $\Theta(x)$ is defined by taking a 2-plane normal to S and containing x: S will intersect this plane in one point and $\Theta(x)$ is defined as the angle around this point, from x to an arbitrary fixed direction in the plane. It follows from 3.15a that for x near S, $\Delta^{vS}(x,y)$ is $\exp[-ip\Theta(x)]$ times a non-singular function, and for y near S it is $\exp[ip\Theta(y)]$ times a non-singular function.

We have found that reduced gauge invariant Green's functions are S independent, we return now to the factor that we divided out, equation 3.4. $< V_p'(C,S) >$ is given by the graph with one photon starting and ending on S, plus the sum of all vacuum bubbles constructed out of the effective propagators and vertices. For example, the determinant term in 3.4 is from the graph which is just one closed loop of the effective propagator Δ^v. As discussed earlier, graphs constructed from the effective propagators and vertices are all S independent. The only S dependence is that which we have found explicitly, the $\delta^2(0)$ term. We can take this term to be an artifact of the way we have defined V_p' when two of the fields in the expansion of the exponential are at the same point, and divide it out of the definition: all Green's functions are then S-independent. It is good that this term can be identified so unambiguously, so that artificial S dependence can be distinguished from a real, physical dependence of the vacuum expectation value of the vortex operator on the area of the minimal surface spanning C. Of the surviving terms in 3.4, the first is exactly the same as the leading term in the expectation value of the Wilson loop, with the replacement of $\frac{2\pi}{e}$ for e. The next is of a form familiar from functional integrals. In fact, $< V_p{}'(C,S) >$ can be interpreted either in the normal way as a functional integral over continuous A and ϕ fields with $V_p{}'(C,S)$ inserted into the integrand, or as a functional integral with no insertion in the integrand but with the A_μ and ϕ fields fixed to have the discontinuities 3.18 on S. Since we want to be careful about divergence problems, we will stay with the first interpretation.

In a Higgs phase, equation 3.16 still holds; there are no extra terms involving tadpoles. To see this, recall that in both the symmetric and Higgs phase, a general connected Green's function is equal simply to the sum of all connected Feynman graphs, without tadpoles but including graphs with "trees". By a tree we mean a piece which connects to no external lines and which can be separated from the rest of the graph by cutting a single propagator, its trunk; we do not mean a graph without loops. The two phases are distinguished by the nature of the sum over all trees, which is equal to the minimum of the effective potential. When we sum over all numbers of vortex photon lines attaching to all Feynman graphs (with trees), we get all graphs (including those with trees) made up of effective vertices and propagators; this is 3.16. The sum over all trees is given by the minimum of the "effective potential in the presence of the vortex", which is obtained from the same graphs as the usual effective potential, but using the effective propagators and vertices.

4. Renormalization of Looplike Operators

In the preceding sections, we neglected renormalization. We did not specify whether quantities were bare or renormalized, we did not include graphs with counterterms, and we did not consider the convergence of the various graphs. These points are the subject of the present section. We include first, as an illustration of some of the ideas, a short section on the renormalization of the Wilson loop operator.

4.1. Renormalization of Green's Functions of the Wilson Loop

The Wilson loop is a composite operator involving products of arbitrarily many elementary fields. The associated divergences, however, turn out to be easily analyzed, at least in the Abelian case: all matrix elements can be made finite by one overall multiplication of the operator[f1]. Gervais and Neveu[16] and Polyakov[17] have shown by the use of elegant methods that the same is true of the non-Abelian Wilson loop.

A general Green's function of the Wilson operator, $< W_e(C) >_{JKL}$, defined by analogy to equation 3.2, is given by

$$< W_e(C) >_{JKL} = \exp \sum_{khmn} \frac{1}{k!h!m!n!} \tag{4.1}$$

$$< [i\oint_C dy_\mu eA_\mu(y)]^k [\int d^4x\ J_\mu A_\mu]^h [\int d^4x\ K\phi]^m [\int d^4x\ L\phi^*]^n >_c$$

The quantity $eA_\mu(y)$ is invariant under renormalization, due to the Ward identity: $e_r A_r = e_o A_o$, where subscripts r and o represent renormalized and bare quantities, respectively. When it is necessary to take a particular renormalization scheme, we will use subtraction at zero momentum (this is acceptable when the charged fields are all massive, which, for simplicity, we assume). If we then take the fields coupled to J, K, and L to be the renormalized ones, the connected Green's functions in equation 4.1 are all renormalized: they are finite when not evaluated at the same spacetime point, and they are integrable over spacetime regions that include coincident points. If we take $J_\mu(x)$, $K(x)$, and $L(x)$ to be smooth, the associated x integrals all exist.

The only possible divergence comes when k is greater than 1, so that there is a multiple integral over C. This may diverge when two or more integrands approach each other along the loop. To emphasize the region in the multiple loop integral where j of the arguments approach each other along the loop, we represent 4.1 as

$$(ie)^j \int_{-\infty}^{\infty} dx \int_C dy_{1\alpha} dy_{2\beta} \cdots dy_{j\gamma}$$

$$\delta(\sum_{i=2}^{j} |y_i - y_1| - x) < A_\alpha(y_1) \cdots A_\gamma(y_j) \ \cdots >_c$$

$$= (ie)^j \int_{-\infty}^{\infty} dx\ x^{j-2} \int_C dy_{1\alpha} dw_{2\beta} \cdots dw_{j\gamma} \tag{4.2}$$

$$\delta(\sum_{i=2}^{j} |w_i| - 1) < A_\alpha(y_1) A_\beta(y_1 + xw_2) \cdots A_\gamma(y_1 + xw_j) \ \cdots >_c$$

where the final ellipsis in the Green's functions indicates the remaining operators, coupled to external sources or to distant points on C. To get the leading $x \to 0$ behavior of the Green's function we use the operator product expansion[18] to write the product of j A's in the Green's function as

$$g_{0\alpha \ldots \gamma}(xw_2, \ldots, xw_j) \cdot 1$$

$$+ g_{1\alpha \ldots \gamma\mu}(xw_2, \ldots, xw_j) \cdot A_\mu(y_1) + \text{higher operators}$$

This form places all of the x dependence in the coefficient functions g_i. We implicitly use a covariant gauge so as to avoid direction dependendent singularities, as found, for instance, in the axial gauge.

At fixed x, the j points cannot be coincident. We cannot assume, however, that the w-integrations converge, as there will be regions in integration space where some subset of the points come together. We will assume a coordinate-space version of Weinberg's theorem[19], which we have not proved but which seems quite plausible: that it suffices to consider just the x-integrations for each subset of points, and if naive power counting indicates that every one of these is convergent then the whole integral will be. In fact this is not necessary: at least when C is an infinite straight line the Green's functions can be written in momentum space. Weinberg's theorem and BPHZ subtraction[18] may then be applied rigorously to verify the conclusions reached below. It is then very plausible that for a smooth curve C the leading divergences are the same as for the straight line. The coordinate-space argument is shorter and perhaps more interesting.

Because A_μ is of dimension 1, at each order in perturbation theory (we are renormalizing order by order) $g_{0\alpha \cdots \gamma}(xw)$ is of order x^{-j} times logarithms and $g_{1\alpha \cdots \mu}(xw)$ is of order x^{-j+1} times logarithms; the higher coefficient functions are all smaller as $x \to 0$. The x-integration associated with the operator 1 is then linearly divergent. Performing the x integration with a distance cutoff Λ^{-1} and the w integration leaving out those subregions where subsets of points become coincident, the c-number piece of 4.2 becomes

$$\{ R(\Lambda) \int_C dy_{1\alpha} n_\alpha(y_1) + \text{finite terms} \} < \cdots >_c \qquad (4.3)$$

where R is of order Λ, $n_\alpha(y_1)$ is a unit vector tangent to C at y_1, and the ellipsis in the matrix element is the same as in 4.2. If C were straight, it would be clear why n_α must appear: there is no other available vector. For C a smooth curve, the leading divergence is the same as for a straight line, since points close together don't see the curvature; therefore $n_\alpha(y_1)$ appears. To put that another way, curvature, the lowest dimensional measure of the actual shape of C, is of dimension two; its coefficient must be two powers less divergent than linear (that is, convergent). This argument fails if C has a kink, as the curvature there is infinite. In general there will be a composite divergence associated with a kink that is one power weaker than the leading divergence, from endpoint effects in the integration. There would therefore be an additional logarithmic divergence for each kink. This was also found by Gervais and Neveu[16]. The integral in 4.3 is simply the perimeter P(C), and so the divergence corresponding to the c-number in the operator product expansion (which arises from graphs with no external lines) can be removed by multiplication of the Wilson loop by an overall factor $\exp[-R(\Lambda)P(C)]$.

The only other divergence is in the coefficient of $A_\mu(x)$. This is logarithmic and is given by graphs which have external lines but are such that cutting a single photon line separates all of the external lines from the Wilson loop. As one might expect, this divergence is actually absent due to the Ward identity. In fact, in our particular scheme (zero momentum subtraction),

$$g_{1\alpha \cdots \gamma\mu}(xw_2,\ldots,xw_j) = G_{\alpha \cdots \gamma\mu}(xw_2,\ldots,xw_j;0) \qquad (4.4)$$

where G is the j-photon Green's function with one leg (the one with index μ) truncated and set to zero momentum. G satisfies a Ward identity

$$k_\mu G_{\alpha \cdots \gamma\mu}(xw_2,\ldots,xw_j;k)=0 \qquad (4.5)$$

From 4.5 and the fact that G is continuous at $k_\mu=0$ (for massive charged particles) it follows that G vanishes at zero momentum. g_1 is identically zero and that there is no divergence coming from connected graphs with external lines. In fact, g_1 can be shown to vanish by gauge invariance in any renormalization scheme.

We find, then, that all matrix elements of the Abelian Wilson loop can be made finite by one overall multiplication.[f2]

4.2. Renormalization of Green's Functions of the Vortex Operator

The analysis of the vortex operator proceeds much like the analysis of the Wilson loop, and the result is the same: an overall multiplication makes all matrix elements finite. There is a potential logarithmic operator divergence, as for the Wilson loop, but its coefficient again turns out to vanish. The vanishing is here more intricate, involving cancellation of field renormalization divergences against composite operator divergences. Our nicest result, that $T[F^{V}_{\mu\rho}]$ is finite for an infinite straight vortex, can be obtained in a few lines (equations 4.10–4.12). The rest of this section is simply power counting to establish that this implies the cutoff independence of all matrix elements of vortex operators for any curve.

We first must ask whether the combinations $\frac{F_{\mu\rho}}{e}$ and $\frac{j_{\mu}}{e}$ appearing in 2.6 and 3.1 refer to the bare or the renormalized fields and charges, since these combinations are not invariant under renormalization. A canonical argument indicates that they must be bare. Equations 2.7 hold only for the bare quantities; if the renormalized quantities were to appear on the left hand side of 2.7, an extra factor of Z_3^{-1} would be needed on the right hand side. In order to have a finite commutator with the Wilson loop, as in 2.3, or with charged fields, as in 2.8, it is then clear that we must have the bare quantities $\frac{F_{o\mu\rho}}{e_o}$ and $\frac{j_{o\mu}}{e_o}$.

We can also reach this same conclusion from the Green's functions. The discussion of the last section did not include graphs with counterterms. This is correct if all fields and couplings, including those in the definition of V_p', are the bare ones (we must have a cutoff at this point). Otherwise, graphs with counterterms enter and spoil the quantization condition. The exponent in the definition of V_p' must therefore be cutoff dependent. If we had started by defining V_p' with the renormalized quantities $F_{r\mu\rho}$ and e_r, we would have found that p/Z_3, not p, was an integer, so that the cutoff dependence would merely be shifted into p. Either way, writing V_p' in terms of cutoff independent p, e, and $F_{\mu\rho}$ gives

$$V_p'(C,S) = \exp\{\frac{\pi p Z_3}{e_r}\int_S d\sigma^y_{\alpha\beta}\tilde{F}_{r\alpha\beta}(y)\} \tag{4.6}$$

with an explicitly cutoff dependent exponent. Note also that this implies that the total coefficient (all counterterms summed) of the one photon graph for $< e_o F_{o\mu\rho} \; V_p'(C,S) >$ or for $< e_r F_{r\mu\rho} \; V_p'(C,S) >$ is πp, with no factors of Z_3. This in turn implies that equation 3.17, which comes entirely from the one photon graph, is correct with $\tilde{F}_{\mu\rho}$ and e either both bare or both renormalized; explicitly,

$$\partial_\mu T< \tilde{F}_{r\mu\rho}(x)V_p'(C,S) > = \frac{2\pi p}{e_r}\oint_C dx'_\rho \delta^4(x-x')< V_p'(C,S) > \tag{3.17'}$$

We now investigate the cutoff dependence of the Green's functions of the vortex operator. Starting with the expression 3.3, we can analyze the divergences of the Green's functions very much as we did for the Wilson loop. There are two differences. One is that V, unlike W, contains a manifest cutoff dependence from field renormalization, as discussed above. The second difference is in the analysis of the composite divergences, where the vortex involves an integral over a two- rather than one-dimensional surface, and the field being integrated is of dimension two rather than one. This second difference is small: because the exponent is dimensionless for both operators, in each case there is only a small number of divergences. The analog of equation 4.2 when j points come together is

$$< V_p'(C,S) >_{JKL} = (\frac{\pi p}{e})^j \int_{-\infty}^{\infty} dx\ x^{2j-3} \int_S d\sigma_{\alpha\beta}^{y_1} d\sigma_{\gamma\delta}^{w_2} \cdots d\sigma_{\mu\rho}^{w_j} \tag{4.7}$$

$$\delta(\sum_{i=2}^{j} |w_i| -1) < \tilde{F}_{\alpha\beta}(y_1)\tilde{F}_{\gamma\delta}(y_1+xw_2)\cdots\tilde{F}_{\mu\rho}(y_1+xw_j)\cdots>_c$$

The small x expansion for the operator product is now

$$h_{0\alpha\beta,\gamma\delta,\ldots,\mu\rho}(xw_2,\ldots,xw_j)\cdot 1 \tag{4.8}$$

$$+ h_{1\alpha\beta,\gamma\delta,\ldots,\mu\rho;\sigma\lambda}(xw_2,\ldots,xw_j)\cdot F_{\sigma\lambda}(y_1)$$

$$+ h_{2\alpha\beta,\gamma\delta,\ldots,\mu\rho}(xw_2,\ldots,xw_j)\cdot\phi^*(y_1)\phi(y_1) + \cdots$$

The expansion includes only gauge invariant operators, because the operator product is gauge invariant. h_0 is of order x^{-2j} and h_1 and h_2 are of order x^{-2j+2}, so that the x-integration for the coefficient of 1 is quadratically divergent, and those for the coefficients of $F_{\mu\rho}$ and $\phi^*\phi$ are logarithmically divergent.

We have neglected the extra operators in $V_p'(C,S)$ that we concluded were needed to assure S-independence. In the two dimensional case, equations 2.16 indicate that the effective dimension of $d^2z\ f_\lambda(z)$ in units of mass is greater than −2 but less than −1. The total dimension of the exponent of Ξ is then negative; it is a "soft" operator. We expect that this property will hold in general. The inclusion of these additional operators, then, will give additional contributions to the coefficient functions in 4.3, but will not lead to any stronger singularities.

The divergence proportional to 1 implies a common infinite factor of the form $\exp[R'(-\Lambda)A(S)]$ in every matrix element of V_p', with

$A(S)$ the area of the surface S, and with R' being of order Λ^2. Just such a divergence was found from the one photon graph in the last section, where it was also shown that such a term, being explicitly S dependent, could not arise from any higher order graph. It can therefore be unambiguously divided out. The operator 1 in the expansion also gives rise to a linearly divergent term proportional to the perimeter $P(C)$, from edge effects in the surface integrals. This was also found from the one photon graph; for this divergence we would expect that higher order graphs will also contribute. At any rate, it can be divided out by a factor of the form $\exp[-K''(\Lambda)P(C)]$, with R'' being of order Λ.

The logarithmic divergence from the h_1 and h_2 can be removed by the addition of three counterterms to the exponent of $V_p'(C,S)$ (as in the case of the Wilson loop, this conclusion can be verified rigorously by use of Weinberg's theorem and BPHZ subtraction when C is a straight line):

$$c_1 = \int_S d\sigma^y_{\alpha\beta}\tilde{F}_{\alpha\beta}(y) \tag{4.9a}$$

$$c_2 = \int_S d\sigma^y_{\alpha\beta}F_{\alpha\beta}(y) \tag{4.9b}$$

$$c_3 = \int_S |d\sigma^y_{\alpha\beta}|\phi^*(y)\phi(y) \tag{4.9c}$$

$|d\sigma_{\alpha\beta}|$ appears in c_3 for the same reason that $dy_{1\alpha}n_\alpha(y_1)$ (which is just $|dy_1|$) appears in 4.3: there is no other Euclidean invariant form for the leading divergence.

The counterterm c_2 is forbidden by by CP invariance. $d\sigma_{\alpha\beta}\tilde{F}_{\alpha\beta}$ is CP even; as a result, so is any logarithmic composite divergence (because a smooth surface is CP invariant to the extent that its curvature can be neglected), while c_2 is CP odd. c_3 is forbidden by the S-independence of gauge invariant Green's functions. The easiest way to see this is to consider a surface S which doubles back on itself. It is important that because of S-independence, there is no new composite divergence associated with this doubling back. The doubled surface cancels out in the definition of $V_p'(C,S)$, because $d\sigma_{\alpha\beta}$ is oriented; it must therefore cancel out in any divergences. It does cancel in c_1 and c_2, but not in c_3, as $|d\sigma_{\alpha\beta}|$ is not oriented. We conclude that all of the composite divergence can be removed by a term of the form $f(\Lambda)\cdot c_1$.

There is still the second source of cutoff dependence in the Green's functions of $V_p'(C,S)$, the factor of Z_3 in 4.6. We see that this is of exactly the same form as the counterterm 4.9a; an appropriate choice of $f(\Lambda)$ can remove this divergence as well, leaving every Green's function of the vortex operator finite.

Since there is only one unknown function of the cutoff, we can determine it by calculating one single Green's function of one particular vortex operator. A convenient choice is

$$T< F_{r\mu\rho}(x)\ V_p'(Z,Y) > / < V_p'(Z,Y) > \qquad (4.10)$$

where Z is the line $y_0=y_1=y_2=0$ and Y is the half-plane $y_0=y_1=0,\ y_2>0$. By Euclidean invariance, this must be given by

$$a(x_z^2)\ \epsilon_{\mu\rho\alpha3} x_{z\alpha} + b(x_z^2)\ [x_{z\mu}\delta_{\rho3} - x_{z\rho}\delta_{\mu3}] \qquad (4.11)$$

where $x_{z\mu}$ is the vector from Z to x which is perpendicular to Z. Under parity, $V_p'(Z,Y)$ goes into $V_p'(Z,Y')$, where Y' is the half-plane $y_0=y_1=0$, $y_2<0$ (note carefully that p doesn't change sign). The gauge invariant Green's functions of these two operators are equal. Under $x_0\to x_0$, $x_i\to -x_i$, the Green's function 4.10 then has natural parity: space-space components are invariant and space-time components change sign. Only the first term is 4.11 has natural parity; it must be that $b(x_z^2)=0$. Equation 3.17′ then requires that

$$a(x_N^2) = -\frac{p}{2e_r}(x_N^2)^{-3/2} \tag{4.12}$$

Remarkably, the Green's function 4.10 is completely determined, and it is cutoff independent without counterterms: $f(\Lambda) = 0$. The factor of Z_3 in 4.6 provides just the cutoff dependence to cancel that in the composite divergence.

Equation 4.12 was found independently by S. Coleman[20] in the context of the non-renormalization of the product $e_o g_o$ where g_o is the magnetic charge of an external monopole. In this context, the operator product arguments are a demonstration that one can in fact make all Green's functions evaluated in the presence of an external monopole finite by renormalization of the magnetic charge.

We illustrate the cancellation with among the order e graphs. The order e contribution to 4.10 is figure 2a. Expanding the effective vertex and propagator in terms of the usual ones gives all graphs with one charged loop and no internal photons, such as those of figure 4. The graphs of figure 4a have the usual divergence; ordinarily this would be cancelled by a counterterm from the order e^{-1} graph, figure 1a. Here, this counterterm is absent owing to the factor of Z_3 in 4.6. However, the graphs of figure 4b are also divergent when the integration over the momenta of the photons attached to V_p' is included. What we have found above is that the divergences of 4b exactly cancel those of 4a. This cancellation has an interesting feature: the different graphs are proportional to different powers of p, depending on how many photon lines attach to V_p'. When we let p vary, the full cancellation occurs only for integral p, as only then can we say that the Green's function is independent of S and that 4.11 is the only allowed form.

The fact that the matrix elements of the vortex operator turn out to be automatically finite is rather important. If we had had to add a counterterm 4.9a to the exponent of V_p', its commutation relations 2.3 and 2.5 would then have become cutoff dependent. We hope to find that there are relations between the commutation relations of V_p' (that is, the fact that it is a vortex operator) and its Green's functions; this would seem unlikely if cutoff independent commutators had been incompatible with cutoff independent Green's functions. This is remeniscent of the situation with Noether currents, generators of exact symmetries: there also we wish to ascribe physical significance to commutators, and there also the commutators and the Green's functions are simultaneously finite.

There is one weakness in the above analysis. What we have really shown with the operator product analysis is that the counterterm c_1 suffices to remove all divergences from Green's functions of V_p when they are expanded <u>order by order in p</u> (thus, in 4.6 we have isolated all graphs of order p^j). It would be preferable if we could first sum to all orders of p, getting effective propagators and vertices as before, and then analyze the divergences directly from the short distance properties of these effective propagators and vertices. We do not expect that our conclusions would change; however, we shall see in the next section that an expansion of the Green's functions in powers of p can sometimes lead to erroneous conclusions, so it would be good to have an analysis of the divergences which did not rely on such an expansion. Note that the finiteness of 4.10 was independent of the expansion in p. It would also be interesting to see what effect hard P- or CP- violating interactions would have on the analysis.[f3]

5. Cluster Properties of Looplike Operators

In this section we give a short discussion of the Wilson[5] and 't Hooft[3] criteria, emphasizing the idea of cluster property rather than vacuum expectation value. The general discussion applies to non-Abelian as well as Abelian theories.

Consider the Euclidean Green's function

$$G(x,C) = \langle \Theta(x)\, W_s(C) \rangle_c$$

$$= \langle \Theta(x)\, W_s(C) \rangle - \langle \Theta(x) \rangle \langle W_s(C) \rangle \qquad (5.1)$$

with $\Theta(x)$ some local gauge invariant operator such as $F^a_{\alpha\beta}F^a_{\gamma\delta}$. In the phases we have mentioned above, this function will behave in one of three ways when C is very large:

Short distance clustering: G(x,C) falls off exponentially with d(x,C), the distance between x and C.

Surface clustering: G(x,C) is nonvanishing near the minimal surface, S_M, spanning C, and falls off exponentially away from that surface.[f4]

Long distance clustering: $G(x,C)$ falls off as a power of $d(x,C)$.

These alternatives may be better understood by considering a typical state of the system in a 3-surface cutting perpendicularly through C. In this 3-surface, one sees a source/antisource pair, c and $\bar{c}$, where C intersects the surface. If the clustering is short range, there are only short range, Yukawa, fields around the sources, and vacuum elsewhere. If the clustering is surface-like, there is a tube of non-vacuum joining c and $\bar{c}$, whose energy per unit length gives rise to a linear potential between the external sources. If the clustering is long range, c and $\bar{c}$ have Coulomb-like fields with a power law fall-off. Long range clustering is only possible if there are massless particles. The other two types of clustering each have a characteristic scale (the range of the Yukawa field or the thickness of the tube) which are determined by the mass m_L of the lightest particle. As $m_L P(C)$ is taken to zero, either by shrinking C or by letting m_L go to zero, the first two cluster properties turn continuously into the third. One might also imagine more general cluster properties, of course. These three, however, seem to cover all those which have arisen in various gauge theories and models.

We can also consider the cluster properties of Green's functions of the vortex operator $V_p(C)$ (p designating the homotopy class); the same three possibilities seem to arise. The cluster property appears to be largely independent of the operator $\Theta(x)$. For instance, if a tube of non-vacuum runs between c and $\bar{c}$, we would expect most local operators to have within the tube an expectation value different from that which they have in vacuum. It may, however, depend on the representation s of the Wilson loop, or the homotopy class p of the vortex operator. A general phase, then, may be characterized by which of the three cluster properties is realized for each representation and for each homotopy class. A confining theory is one in which the Wilson loop, at least in some representations, has surface clustering, as this implies linear confinement of external charges in those representations; this is the Wilson criterion[5]. The 't Hooft criterion defines a (completely broken) Higgs theory as one in which some of the vortex operators have surface clustering, as this implies that magnetic flux is forming into tubes.

This classification is closely related to the usual classification of phases in terms of the vacuum expectation values of the Wilson and vortex operators. For a very large curve C, the vacuum expectation value of a general looplike operator X(C) will be dominated by $\exp[-S_{cl}]$, where

$$S_{cl} = \int d^4x < L(x)\ X(C) >_c / < X(C) > \tag{5.2}$$

and L(x) is the Lagrangian density. The connected Green's function in 5.2 is a special case of 5.1. For short distance clustering the integrand in 5.2 will be nonzero only for x near C, so that the whole integral is proportional to P(C): < X(C) > follows a perimeter law. For surface clustering, the integrand will be nonzero only for x near the minimal surface, so that the whole integral is proportional to $A(S_M)$. In phases without massless particles, there is a one-to-one correspondence between an area law and surface clustering, and between a perimeter law and short range clustering.

For long range clustering, the integrand is proportional to $d(x,C)^k$; the behavior of the integral depends on the particular value of k. In QED k is -4; the integral is dominated by small values of d(x,C) and is proportional to P. If k were -2, the integral over d(x,C) diverges linearly until cut off at the linear dimension of the loop; the whole integral would be proportional to the square of this linear dimension, as it is for surface clustering. In this case there also be a linear potential between external sources, but without the formation of a flux tube. There would be, however, a strongly interacting massless particle, which is not observed. Furthermore, it is not clear that it is possible to find a consistent physical picture in which k is -2.

One of 't Hooft's restrictions on the possible phases is that if a Wilson loop and vortex operator do not commute (the phase z_{sL} in 2.2 is not 1) they cannot both have short range clustering[3]. The key idea is to consider the Euclidean Green's function

$$< V_p(C,S)\ W_s(C') > \tag{5.3}$$

for two large curves C and C'. If both operators have short range clustering only, and if C and C' are not near each other at any point, the Green's function 5.3 should be invariant under translation of C', as this is essentially a translation through vacuum, except when C' crosses S. When C' does cross S, the Green's function jumps by a phase due to the canonical commutator. A short four-dimensional argument then shows this to be inconsistent with the single-valuedness of the Green's function; see reference 3. 't Hooft mentions a phase ambiguity in the Green's function 5.3; this is the fact that it can change by a phase under a change of S with C and C′ fixed. It is important that once S is fixed, the Green′s function is not ambiguous and must be single-valued.

Short distance clustering is analogous to that for pointlike fields in a massive theory, where the general connected two-point Green′s function falls off exponentially with distance. Long distance clustering is analogous to that for pointlike fields in a massless theory. Surface clustering is a new feature; it seems to arise when there is a flux which can neither spread nor be shielded. To learn more about it we shall consider a simple example of an operator with surface clustering, namely the vortex operator in a Higgs phase.

In an Abelian theory without magnetic monopole fields, we can show that the vortex operator can never have short range clustering, so that in any Abelian theory without massless particles it will have a surface clustering and obey an area law. From 3.17′ and Gauss′s law we find

$$\int_B d\sigma^x_{\mu\rho}\; T< \tilde{F}_{\mu\rho}(x)\; V_p{}'(C,S) > = \frac{4\pi p}{e} < V_p{}'(C,S) > \qquad (5.4)$$

where C is a large curve and B is a 2-sphere linking C (in four dimensions curves and 2-spheres link). All fields and charges in this section are taken to be renormalized. The integral over B is independent of the radius of B. On the other hand, since the Green′s function in 5.4 is gauge invariant, short range clustering would require it to fall exponentially when the radius of B is greater than m_L; this is inconsistent with equation 5.4.

What we have shown is really quite simple: magnetic flux can never be shielded. If the absence of massless fields then makes it impossible to have a Coulomb field, magnetic flux can only form into tubes. Given that magnetic flux is confined, one can extend 't Hooft's more recent results[4] to the Abelian case to show that all Wilson loops must obey a perimeter law. It follows that a continuum Abelian theory (without magnetic monopole fields) never confines. This result was anticipated by Mandelstam[2] on the basis of the existence of the Abelian Coulomb gauge.

One might try to argue in a different way that the Green's functions of the vortex operator had to be short ranged in a phase without massless particles. Take a large curve C, with the surface S far away from the minimal surface. Consider

$$< \Theta(x)\ V_p'(C,S) >_c \ / < V_p'(C,S) >$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} < \Theta(x)\ [\frac{\pi p}{e}\int_S d\sigma^y_{\alpha\beta} \tilde{F}_{\alpha\beta}(y)]^k >_c \qquad (5.5)$$

where again $\Theta(x)$ is any gauge invariant operator and where x is a point near the minimal surface of C but far from C itself. Because there are no massless particles, every Green's function in 5.5 falls exponentially for x distant from S. In particular they are vanishingly small when x is on the minimal surface, so that surface clustering is impossible. Further, since $< \Theta(x)\ V_p'(C,S) >_c$ is exponentialy small for x far from S, and also independent of S, it is exponentially small except for x near C. This is in direct disagreement with what was shown above. The problem must lie in the expansion 5.5: while this expansion is formally correct, the long distance behavior of the sum is not the same as that of the individual terms. This is the source of our statement, at the end of the section on renormalization, that the expansion in powers of p is not always to be trusted.

We would like to see the surface-like cluster property emerge from the earlier graphical expansion. Consider the expectation value of the Higgs field in the presence of the vortex operator:

$$\phi^{VS}(x) = \langle \phi(x)\, V_p'(C,S) \rangle / \langle V_p'(C,S) \rangle \qquad (5.6)$$

where $\phi(x)$ is the Higgs field, C is a large curve and S is taken, for convenience, to lie far away from the minimal surface of C. In a 3-surface which is perpendicular to C and which cuts it at two points we have figure 5. Near the small loop 1, which is far from C and from its minimal surface, gauge invariant connected Green's functions will vanish and $\phi^{VS}(x)$ will be position dependent but its values will lie in the set M of minima of the Higgs potential. For the Abelian theory M is the set of complex numbers of modulus u, where u is the vacuum expectation value of the Higgs field, but the present argument generalizes readily to a non-Abelian theory.

If we have an Abelian theory, we know from 3.18 and 3.19b that $\phi^{VS}(x)$ will be $u \cdot \exp[ip\Theta]$ as we traverse the infinitesimal loop 1 and Θ goes from 0 to 2π. On the loop 1, $\phi^{VS}(x)$ is seen to describe an element of the homotopy group $\Pi_1(M)$ identical to the element of $\Pi_1(G)$ associated with the vortex operator. We may now imagine enlarging the loop, sliding it off S, taking it to the position of loop 2 and shrinking it to a point, without ever getting close to c or $\bar{c}$. Because the fields are singular only at S, $\phi^{VS}(x)$ must be essentially constant on loop 2: it maps out an element of the trivial homotopy class. By definition there is no way to continuously deform an element of one homotopy class into an element of another while staying within M. It follows that, somewhere between 1 and 2, $\phi^{vS}(x)$ took values outside of M; this must be the case at least within a tube between c and $\bar{c}$, and therefore on a surface spanning C. This is precisely the argument by which one shows that when the Higgs field at spatial infinity in two space dimensions maps out a non-trivial element of the homotopy group, there must be a "lump", a Nielsen-Olesen vortex[8,9], somewhere in space. The tube between c and $\bar{c}$ is a Nielsen-Olesen vortex.

Where $\phi^{vS}(x)$ does not lie in M, gauge invariant connected Green's functions such as that for the Higgs potential will be non-vanishing. It follows that the vortex operator in a completely broken small coupling Higgs phase has surface clustering. This is the source of the 't Hooft criterion. One may also check, order by order, that the Green's functions of the Wilson loop are short range, because all fields are massive. $\phi^{vS}(x)$ is given by all trees whose trunk is a boson propagator. Each individual graph, by the earlier argument, has short range clustering, but the sum, as given by the minimum of the effective potential in the presence of the vortex, has surface clustering.

6. Conclusions

We have dealt mainly with technical aspects of the Abelian vortex operator. Our two dimensional "Dirac lump", and our solution to the Dirac veto problem, are probably more amusing than they are useful, at least for the present problem where the methods of Wu and

Yang can be used. For a field theory of electrons and monopoles, it may be helpful to use ideas akin to ours, as the monopole is no longer classical and the method of writing a monopole field theory as a sum over monopole paths is rather formal.

We have shown that the divergences of looplike composite operators can be analyzed in a straightforward way by use of the operator product expansion. We believe that any attempt to obtain equations of motion for looplike operators (Wilson loops or vortex operators) must include a careful treatment of short distance questions, along these lines. Also, such questions as the existence of the limit in Mandelstam's construction of the dual Wilson loop[2] can probably be analyzed with these methods.

We emphasized the cluster properties of looplike operators because they provide a more detailed physical picture than simply the vacuum expectation value. We saw two correct ways (divergence equation and tree sum) to find the surface-like cluster property in an Abelian Higgs phase, and one incorrect way (expansion in p).

Acknowledgements

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FOOTNOTE

f1) I would like to thank Professor K. Bardakci for helpful discussions. The work of section 4.1 was done before the appearance of references 16 and 17.

NOTES ADDED IN PROOF

f2) Products of the form W(C)O(x) with O(x) a local operator and x lying on C arise when one attempts to derive equations of motion for W(C). The divergences of such products may be easily analyzed with the methods developed here; they require counterterms having the form of similar products involving local operators of dimension less than or equal to that of O(x).

f3) The result $b(x_z^2)=0$ may also be derived using CP-invariance rather than P. If parity and CP are both violated by hard (dimension 4) interactions, a counterterm of the form c_2 will be needed, but it can still be shown that c_1 is unnecessary. The operator with finite Green's functions is then a vortex operator times a Wilson loop of charge determined by the magnitude of P and CP violation. This would suggest that with hard P/CP violation, a theory of Dirac monopoles is not renormalizeable but a theory of Dirac dyons is.

f4) Recent work on the roughening transition[21,22] shows that when there is surface clustering the Green's function 5.1 actually spreads out over a distance from the minimal surface which becomes infinite when the linear size of the loop C becomes infinite (though the ratio of the spreading width to the linear size goes to zero). This occurs because the flux tube, though finite in thickness, fluctuates in position. The discussion following equation 5.2 is unaltered, as the action is proportional to at least the area of the minimal surface for every configuration of the fluctuating tube.

REFERENCES

1) S. Mandelstam, Phys. Lett. 53B (1975) 476

2) S. Mandelstam, Phys. Rev. D19 (1979) 2391

3) G. 't Hooft, Nucl. Phys. B138 (1978) 1

4) G. 't Hooft, Nucl. Phys. B153 (1979) 141

5) K. Wilson, Phys. Rev. D10 (1974) 2445

6) R. Savit, Rev. Mod. Phys. 52 (1980) 453

7) R. Brandt, F. Neri, and D. Zwanziger, Phys. Rev. D19 (1979) 1153

8) S. Coleman, in New Phenomena in Subnuclear Physics, ed. A. Zichichi (Plenum, New York, 1977), p. 297

P. Goddard and D. I. Olive, Rep. Prog. Phys. 41 (1978) 1357

9) H. B. Nielsen and P. Olesen, Nucl. Phys. B61 (1973) 45

10) S. Coleman, Phys. Rev. D11 (1975) 2088

11) K. Nishijima, Fields and Particles (Benjamin, New York, 1969) section 5-5.

12) R. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965)

13) I. Gelfand and A. Yaglom, J. Math. Phys. 1 (1960) 48

14) T. T. Wu and C. N. Yang, Phys. Rev. D12 (1975) 3845

15) J. C. Taylor, Gauge Theories of Weak Interactions (Cambridge, Cambridge, 1976)

16) J. Gervais and A. Neveu, Nucl. Phys. B163 (1980) 189

17) A. M. Polyakov, Nucl. Phys. B164 (1980) 171

18) R. Zimmerman, in Lectures on Elementary Particles and Quantum Field Theory, ed. S. Deser, M. Grisaru, and H. Pendleton (M.I.T., Cambridge, 1970), vol. 1

19) S. Weinberg, Phys. Rev. 118 (1960) 838

20) S. Coleman, U. C. Berkeley seminar.

21) M. Luscher, DESY preprint 80/97, 1980

22) J. M. Drouffe and J. B. Zuber, Saclay preprint DPh-T-80/120, 1980

FIGURE CAPTIONS

1. a. The graph for $A_{\mu}^{vS}(x)$. The double line represents the surface S.

   b. A typical graph for G(P,C,S). The curved line is the path P of the line integral; the straight segments are scalar propagators.

2. a. The order e graph for $< A_{\mu}(x)\ V_p{}'(C,S) > / < V_p{}'(C,S) >$. The heavily circled v's are effective vertices, the lightly circled v's indicate effective propagators.

   b. An order e^3 graph for the same matrix element.

3. a. The discontinuous graph for $< V_p{}'(C,S)\ W_q(C') >$. The single heavy line represents C'.

   b. Another graph for the same Green's function.

   c. Another graph, connected to C' by three photons.

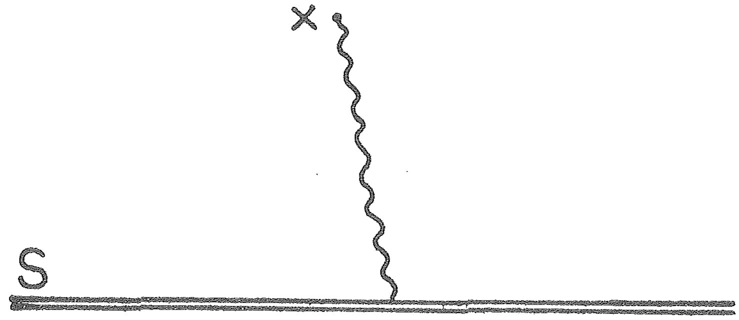
4. Graphs in the expansion of figure 2a.

   a. The two graphs with field renormalization divergences.

   b. Two of the graphs which have composite divergences.

5. The Green's function 5.6, considered in a 3-surface. c and $\bar{c}$ are the intersections of C with the 3-surface; s is the intersection of S with the 3-surface. The function 3.3 maps loop 1 into a nontrivial path in M; it maps loop 2 into a trivial path.

a)



b)

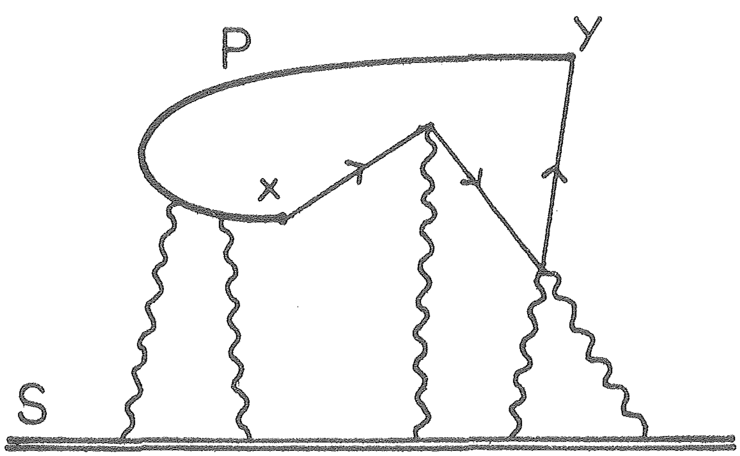


Figure 1

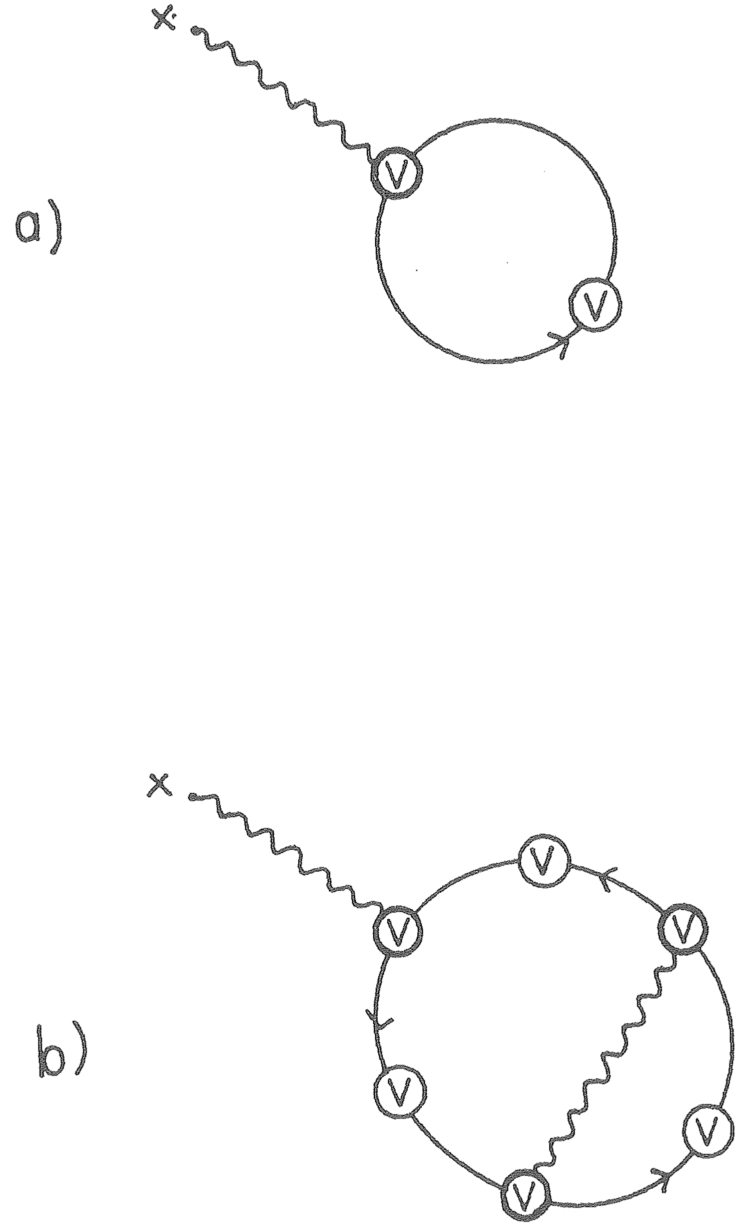
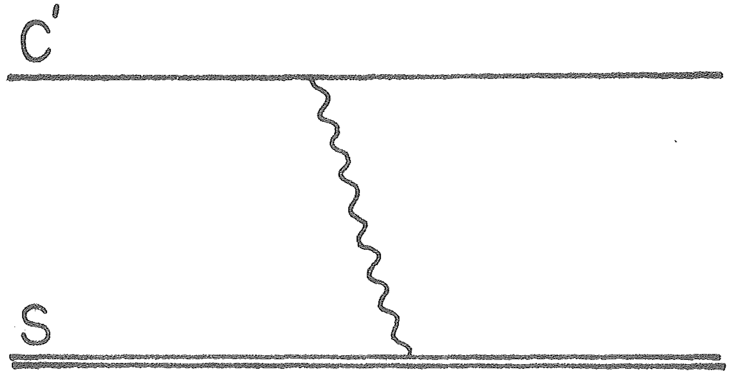


Figure 2

a)



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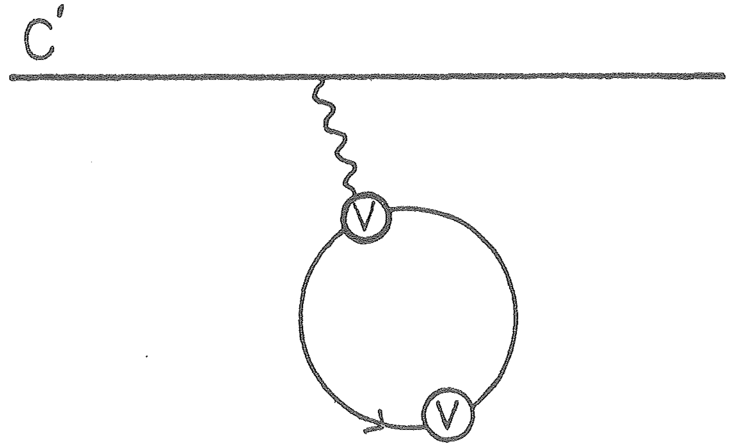

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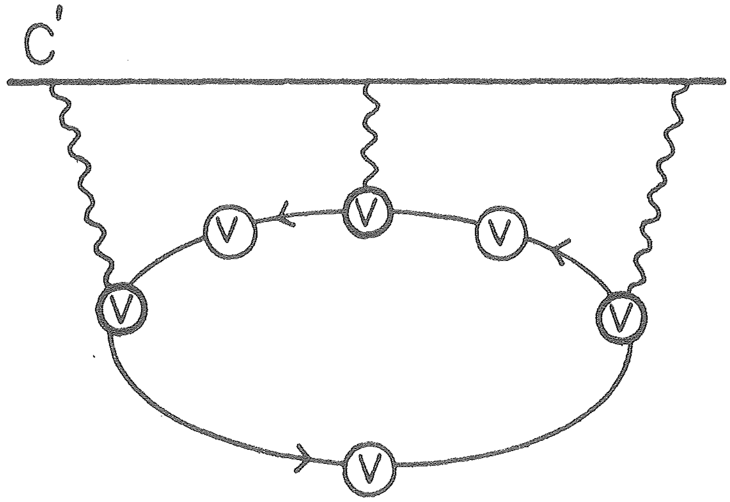

Figure 3

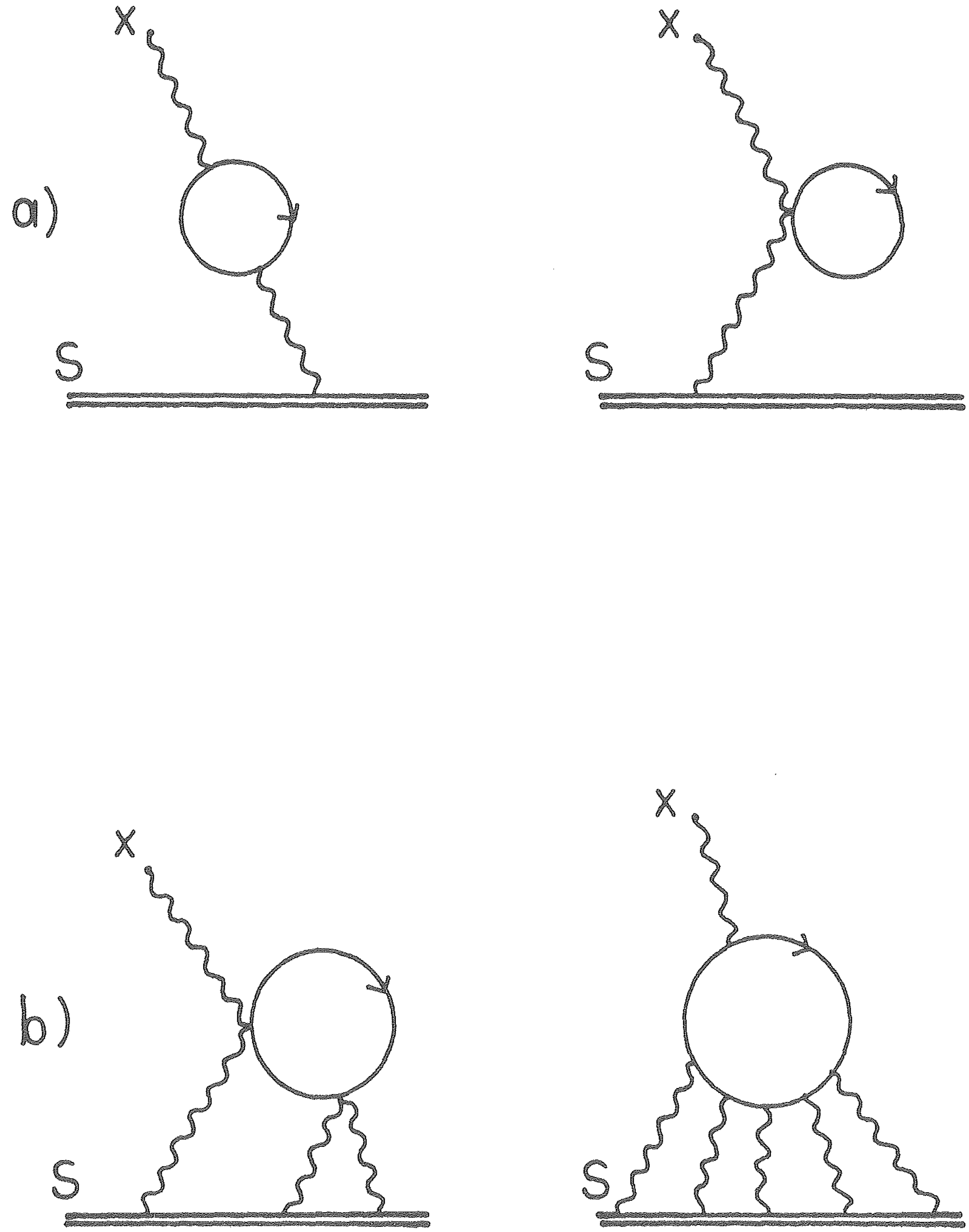

Figure 4

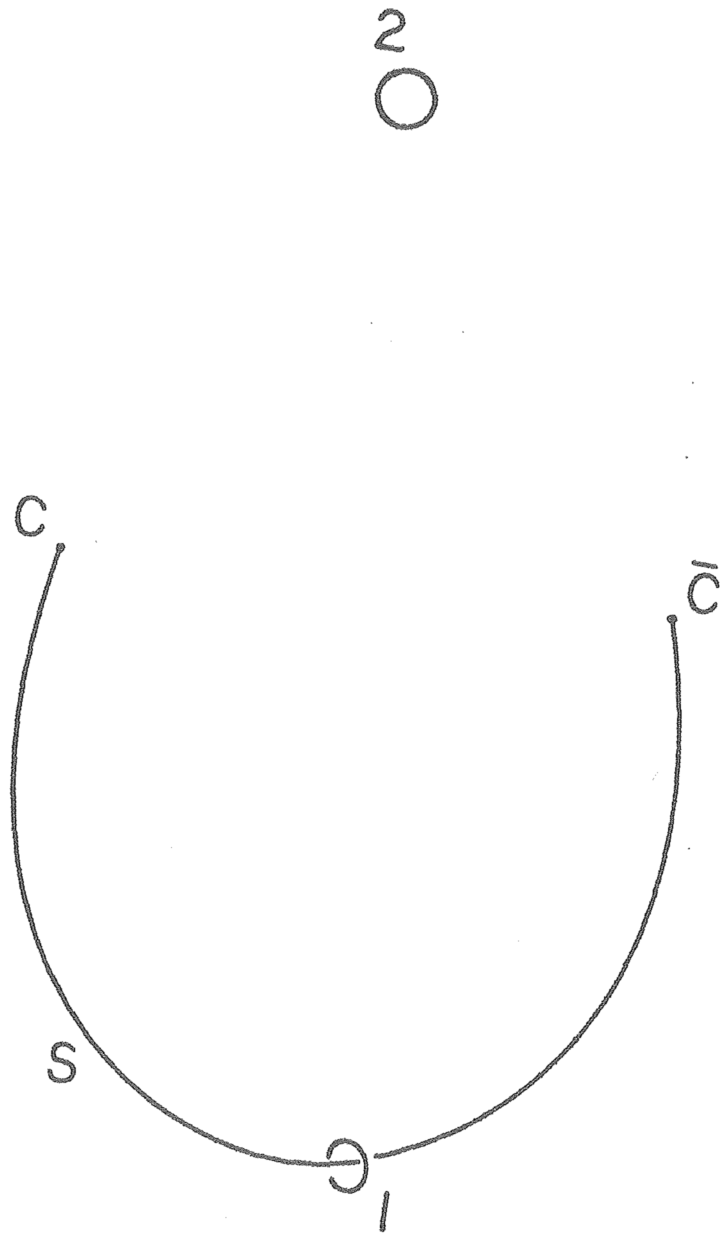

Figure 5